Art. IX.—*Improvements in Fundamental Ideas and Elementary Theorems of Geometry.* By Martin Gardiner, Esq., C.E.

[Read before the Institute 13th July and 3rd August, 1859.]

Many eminent mathematicians have decided on the necessity of introducing a systematic motional philosophy into the higher Geometry.

Monsieur M. A. Chaucy, as far back as 1846, brought the subject before the Academy of Sciences of Paris, in a paper entitled " Memoire sur les avantages que présente dans la Géométrie Analytique l'emploie de facteurs propres á indiquer le sens dans lequel s'effectuent certaines mouvements de rotations, et sur les résultats construites avec les cosinus des angles que deux systemes d'axes forment entre eux."

In this memoir, after explaining what has been done in his algebra as to the legitimate restrictions of notation, he says:—" This expedient, already generally adopted by geometers, has caused the disappearance of the uncertainty which the interpretations of certain formulas presented, and the contradictions which symbolical calculations seemed to present."

Dr. August Wiegand, professor in the University of Halle, in a note to the editors of the *Mathematician,* in 1847, entitled " Generalization of the leading operations of Arithmetic in reference to Geometry," shows, by a motion of rotation, how we may find rational interpretations for what are known as imaginary expressions.

Something of a similar method is given in De Morgan's *Double Algebra,* published in 1849; and more extended ideas concerning these and kindred subjects are the ground-work of Sir William Hamilton's celebrated *Quaternions.*

Writers on Trigonometry have indeed been universally compelled to take cognizance of directions on straight lines and formations of angular magnitudes; and Professor Chasles, in his " Géométrie Superièure," shows, beyond all doubt, that great advantages result from introducing such conceptions into pure geometrical reasonings.

All this is well known to mathematicians, and many are of opinion that a more liberal motional philosophy should be formally introduced into the common primary elements. This, to some persons, may appear an unnecessary innovation;

but, laying aside its intrinsic worth in developing the fundamental ideas of the science, there are other reasons of great importance why it should be introduced; for by its means only we are enabled, in some cases, to distinguish between what is or what is not a perfect geometrical proposition. To show this I need only instance the following enunciation:—"Given two straight lines MM NN in position, and the points PQ, one in each line; through two given points BC to draw two straight lines BO CO, making the angle BOC of a given magnitude θ, and such that EF, being the respective points in which BO and CO cut MM and NN, we shall have the segment PE to the segment QF in a given ratio $m' : n'$."

Here, by not taking cognizance of the method of formation of the angular magnitude θ, and of the formation of the segments as to direction on the given lines, and of the sign of the ratio, we have in reality four distinct propositions under one enunciation. The result is that, to what is meant by this enunciation, it is absolutely impossible to give a direct general solution applicable to all particular cases, from which the determination of the 'limits' and 'porismatic' relations can be adduced.

Now the Géométrie Superièure is almost entirely dependent on homographic pencils and divisions, and their peculiar developments—all founded on improved ideas. It has not been found compatible with its principles to use the geometrical truths transmitted to us by the Greeks, as many of these truths have not sufficient generality and precision. But it is possible to improve the ordinary geometry, and give to it all the advantages claimed as the distinguishing characteristic of modern theories: all that is required to effect this is a skilful introduction of conceptions of motion into its fundamental ideas and theorems.

Whether a comprehensive motional philosophy (embracing movements of rotation, translation, &c.) was ever used in the writings of the Greeks, it is hard to determine: most probably it was the groundwork of Euclid's porismatic or second elements. But, from the accounts of the method pursued by Apollonius in his books of solutions, it is evident this great geometer did not make use of any elements established on such principles: for, according to Pappus, the general problems were divided into numerous particular cases, each case receiving separate analysis, composition, and discussion, "just," he says, "as variations in the figure might require."

However, there are conflicting opinions as to the cause of this peculiar piece-meal method adopted by Apollonius. Mons. M. Chasles—after giving a general solution to the Determinate Section, and finding expressions for the limits of the involved ratio—makes the following remarks, bearing on the subject:—" La recherche de ces expressions offrait surtout des grandes difficultés qui ont exigé toutes les resources du Géomètre. Car Apollonius ne connaissant pas les différentes relations analytiques de l'involution dont nous avons fait usage, c'est au moyen de figures, et par des considerations de pure géométrie, différentes dans les trois cas, qu'il est parvenu à la détermination des valeurs en question." The geometers of the English school, having unfortunately followed in the footsteps of Apollonius, were unable to perceive the cause which led to the necessity of having recourse to supplementary figures.

But, that I am right in attributing the method to the want of precision and generality in the language employed to express the elementary operations and theorems, will be evident from the solutions which I have effected, independent of involution and homographic theories, for the celebrated problems of the Greek and French schools.

These solutions I intend to present immediately to the Institute, with critical and historical notes.

DEFINITIONS.

A and B representing points, it is to be remembered that when we say 'line AB,' or simply AB, we mean the distance between A and B understood as being described from A direct to B; and that when we say 'line BA,' or 'BA,' we mean the distance between these same points, as described from B direct to A.

The two directions along a straight line are said to be the primitive directions of the line.

To distinguish between the two directions along a straight line, we call one of them 'left,' and the opposite one 'right.'

The 'relative' magnitude and distance of a finite straight line BB, in respect to any other straight line CC, is the magnitude and direction on this other line CC, measured direct from the projection of its first point B to that of its second point B.

If a straight line AA be cut perpendicularly by another BB, then, looking from the point of intersection along the right direction on AA, and having the point of intersection nearest the eye, we say the direction on the perpendicular, from the point of intersection towards our right hand is 'right normal,' and that the opposite direction on the perpendicular is 'left normal.'

Directions are said to be 'oblique right' or 'oblique left,' in respect to the primitive directions on another straight line AA, just according as (when taken from the points in which they cut this line) they lie between a normal and a right primitive, or between a normal and a left primitive.

If a straight line, which we may conceive produced to infinity in its primitive directions, be supposed to become rigid, and one point of it to be permanently fixed, the rigid line being otherwise capable of movement in any plane in which it may lie, then it is evident there are but two ways of revolving the line in this plane—one being 'right rotation,' and the same as that in which the hands of a watch would move if the dial-plate were towards us, and in the plane; the other, or contrary, is called 'left rotation.'

If AA and BB be two straight lines, and I their point of intersection, then the 'angle IA right to B,' means the angle formed at I by a rigid line having I as a fixed pivot, and revolving from a position in AA by a right rotation until its first arrival into the position BB, the revolving line being supposed produced indefinitely on both sides of the pivot.

If AA and BB be two straight lines, and I their point of intersection, then the 'angle IA left to B' means the angle formed at I by a rigid straight line having I as a fixed pivot, and revolving by a left rotation from a position in AA until its first arrival into the position BB, the revolving line being supposed produced indefinitely on both sides of the pivot.

The 'angle AA right to BB,' means the same as angle IA right to B. The 'angle AA left to BB,' means the same as angle IA left to B.

If AA and BB be two straight lines, and I their point of intersection, then 'angle IA right round to IB' means the angle formed at I by a straight line having one of its extremities in this point, revolved by right rotation from the actual direction IA until it arrives in the actual direction IB.

If AA and BB be two straight lines, and I their point of intersection, then 'angle IA left round to IB,' means the angle formed at I by a straight line having one of its extremities in this point, revolved by left rotation from the actual direction IA until it arrives in the actual direction IB.

If AA and BB be two straight lines, and I their point of intersection, the angle 'right AB' means the angle IA right round to IB, and the angle 'left AB' means the angle IA left round to IB.

The angle (AB) means either one or the other of these last two.

Similar remarks, as far as regards right or left formation, apply to the arc of any curve: the elements of the arc being supposed described by their respective radii of curvature.

THEOREMS.

Magnitudes of like formations, whether lines or angles, should receive like signs, viz.:—

All magnitudes of right formations should receive like signs, and all magnitudes of left formations should be distinguished by other like signs.

Magnitudes of opposite formations should receive opposite signs, viz.:—

Right formations should receive opposite signs to left formations.

Magnitudes of the same 'relative' formations, such as 'left oblique,' should receive like signs.

Magnitudes of opposite 'relative' formations, such as 'right oblique' and 'left oblique,' should receive opposite signs.

Numbers, lines, or other magnitudes, whose ratios truly express the ratios of magnitudes of like or unlike formations, must accordingly be of like and unlike signs.

If two straight lines MM MN intersect each other, then the following relations exist amongst the angles at their point of intersection:—

The angles MM right to NN are equal to each other.

The angles NN right to MM are equal to each other.

The angles MM left to NN are equal to each other.

The angles NN right to MM are equal to each other.

The sum of the two angles MM right to NN, and NN right to MM = half revolution right.

The sum of the two angles MM left to NN, and NN left to MM = half revolution left.

If we have any number of points, and that we assume any one of them as a starting-point, then, in respect to any primitive direction whatever, will the sum of the relative distances from the starting point to any of the others, and from this last to another, and so on consecutively, as may be, until a final arrival at the starting point be equal zero. Cor.

If $a\ b\ c$ be any three points in a straight line, then will $ab + bc + ca =$ zero.

And from this we can generalise Euclid's Second Book.

Given any number of straight lines AA, BB, CC, DD, EE, FF, &c., if one of each of the following pairs of angles be given, viz.:—Either AA right or left to BB, BB right or left to CC, CC right or left to DD, DD right or left to EE, EE right or left to FF, &c., then BB and EE being any two of these lines, the angles BB right and left to DD will be given.

If BB and AA be parallel straight lines, and CC any straight line intersecting them, then will the angle CC right to AA = angle CC right to BB, and angle CC left to AA = CC left to BB, and angle AA right to CC = BB right to CC, and AA left to CC = BB left to CC. And reciprocally, if any one of these relations exists amongst the angles made by CC with BB and AA, then will BB and AA be parallels.

If there be any two intersecting straight lines, and that on each of them there is a pair of points, such that the rectangle under the distances of the point of intersection from the points on the first line shall have the same magnitude and sign as the rectangle under the distances of the intersection from the points on the second line, then will these two pair of points lie in the circumference of a circle. And reciprocally, &c.

If ABCD be any four points so related that the angle AC right to B = angle DC right to B, or that AC left to B = DC left to B, then will these four points lie in the circumference of a circle. And reciprocally, &c.

If ABCD be four points such that angle AB right or left to D is equal angle CB right or left to A, then will the line AD touch the circle ACB in A. And reciprocally, &c.

If BA be fixed points, and C a point so restricted that the angle CA right or left to B is of a constant magnitude, then will the locus of C be the circumference of a fixed circle through the points A and B. And the reciprocal is also true.

If *aa* and *bb* are fixed straight lines through fixed points A and B, should *cc* and *dd* be any other pair of straight lines through A and B, making the angle *cc* right or left to *aa* equal to the angle *dd* right or left to *bb*, then will the locus of the intersection of *cc* and *dd* be a fixed circle passing through A and B.

If pp and qq be fixed straight lines, and P and Q fixed points, and that mm and nn are fixed straight lines and M and N fixed points; if straight lines $m'm'$ and $n'n'$ passing respectively through M and N make the angles $m'm'$ right to mm, and $n'n'$ right to nn, respectively equal to the angles $p'p'$ right to pp and $q'q'$ right to qq—where $p'p'$ and $q'q'$ are any two straight lines through P and Q, intersecting in the circumference of a fixed circle through P and Q—then will the locus of the intersection of $m'm'$ and $n'n'$ be a fixed circle passing through M and N.

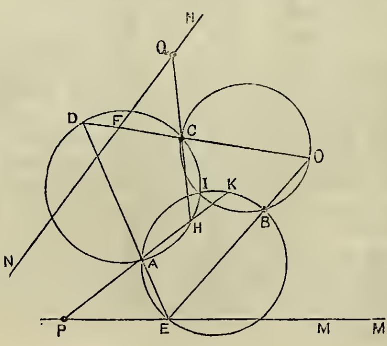
And if the angles mm right to $m'm'$, and nn right to $n'n'$ be respectively equal to the angles $p'p'$ right to pp and $q'q'$ right to qq; the locus of the intersection of mm and $m'm'$ is a fixed circle passing through M and N. See "Note" at end of theorems.

If a circle always pass through two points, should its radius become infinite, the straight line through these two points lies in its circumference.

Any straight line can be regarded as an arc of an infinitely great circle, and if we know on which side of the line the infinitely distant centre is, we then know the right and left directions on the line, &c.

If any hypotheses causes two points in a straight line or in the circumference of a circle to coincide, then if these points have been always or not necessarily always situated in the line or circle, so accordingly will any other circle or straight line through these points touch or coincide with this line or circle.

PRACTICAL APPLICATIONS OF THE PRINCIPLES CONTAINED IN THE PRECEDING PAGES.



Given two straight lines MM NN *in position, and the point* P *in* MM, *and* Q *in* NN; *through two given points* BC *to draw two lines* BO CO, *making the angle* OB *right to* C *equal a given angular magnitude* θ *right, and such that* E *and* F *being the respective points in which* OB *and* OC *cut* MM *and* NN, *we shall have* PE *to* QR *in the given ratio* m *to* n.

ANALYSIS.

Draw QC. Suppose we draw PA making angle PA right to E = angle QC right to F, and PA : QC : : PE to QF; then the triangle APE is evidently similar to CQF, and angle AP right to E = angle CQ right to F, and therefore, H and D being the points in which PA and EA cut QC and FC, a circle can pass through the points ACDH. It is also evident the points AH and circle AHC are given. Since angle OB right to C is given in magnitude, and B and C in position, the circle OBC is given; and therefore also the point I in which it again cuts circle AHC. Again, the angle AI right to E or D being equal to the angle CI right to D or O, it is equal angle BI right to O or E: therefore a circle can pass through AIBE; but AIB are given points; hence this circle is given, and the point E in which it cuts MM, and therefore also the lines EBO OCF.

COMPOSITION.

Draw QC; through P draw PA making angle PA right to M = QC right to N, and on it take PA, such that PA : QC :: $m : n$; through C A and the point H in which PA cuts QC, describe a circle; through B and C describe that circle such that an angle at any point of its circumference, from B right to C, shall be equal angle θ right; through the points A B and the other point I in which this circle cuts circle AHC, describe a circle; through either point E in which circle ABI cuts MM, draw EB to cut circle CIB in O; draw OC to cut NN in F: then will EBO, FCO, be as required. For let D be the point of intersection of EA and FC.

The angle AE or AD right to I = angle BE or BO right to I = CO or CD right to I; and therefore the point D is in circumference AHC. The angle AD or AE right to H or P = CD or CF right to H or Q; and since PA right to E = to QC right to F, the triangles PAE QCF give as PE : QF :: PA : QC :: $m : n$.

DISCUSSION.

If the signs of the directions on MM and NN are known, it is evident that, according as the sign of $\frac{m}{n}$ is positive or negative, so must the directions PA and QB be like or unlike when regarded as relative to directions of same species on MM and NN respectively; it is also plain that if $\frac{m}{n}$ be restricted to one sign, there is but one answerable point A, one circle AIB, two points E, and two solutions to the problem.

But if $\frac{m}{n}$ be unrestricted as to sign, there are two answerable points A, two circles AIB, and therefore four answerable points E, and four solutions. Moreover, since the two points A must be on opposite sides of MM, one of the circles AIB will always cut MM in real points E, the other two points E being real or imaginary, just according as the other circle AIB cuts MM in real or imaginary points.

We will now find the limiting values for the angular magnitude θ right, and for the ratio $\frac{m}{n}$, so as to be enabled so say 'a priori' when the points E are real, imaginary, &c.

Limiting values for 'θ right.'

To find the limiting values for the angles 'θ right,' when the rest of the data is unchangeable, we may proceed as follows:—

The point A is evidently fixed independent of θ right. Looking on the triangle DEO, we see the angle DO right to E is constant for all values of θ, and it is evident that when OE right to D or OB right to C is at its limit, then will ED right to O or EA right to B be at its limit; but this last angle is evidently at a limit when the circle BAE touches MM. Hence it is evident that, by describing the two circles through A and B, which touch MM, and putting i and i for the other points in which they cut circle ACH, then will the angles i B right to C, and i B right to C, be the required limits. And it is moreover evident that, according as the given magnitude of 'θ right' is not comprehended between these limits, or equal to one of them, or comprehended between them, so accordingly will the corresponding circle AIB cut MM in two imaginary, in two real and co-incident, or in two real and distinct points, E. It is also evident that should A and B not lie on the same side of MM, then will the points E be real for all values of 'θ right.'

Limits for the ration $\frac{m}{n}$.

To find the limiting values for the ratio $\frac{m}{n}$, when the rest of the data is unchangeable, we may proceed as follows :—

The points C and H are fixed independent of the variations of $\frac{m}{n}$. And our object is to determine the limiting positions for A on PA (which is fixed in position) for then will the resulting values of $\frac{PA}{QC}$ be the limiting values required. Now the angle EA right to B being evidently constant, it follows that the other point K, in which PA again cuts circle AEB, is constant, no matter how A may vary; and therefore putting a and a for the two points in which the two circles through B and K to touch MM again cut PA, it is evident that $\frac{Pa}{QC}$ and $\frac{Pa}{QC}$ are the required limits. It is clear that these limiting ratios have the same sign, and that according as any value of $\frac{m}{n}$ (having like sign with these limits) is of a magnitude comprehended between them—equal to one of them—or not comprehended between them, so accordingly will the corresponding circle AIKB cut MM in two imaginary, in two real and coincident, or in two real and distinct points, E. When $\frac{m}{n}$ and the limiting ratios have different signs, the points E are always real. If B and K be not on the same side of MM, the limits are imaginary, and the points E real for all possible values of $\frac{m}{n}$.

Remarks on the consequences arising from supposing particular positions and magnitudes for the involved data.

If the point B coincides with C, then the circle COB is infinitely small, and the points I and O are coincident with C. It is also evident that the direction CI is in this case a tangent to the circle AHC at C, and that the direction BI makes the angle IB right to C = angle 'θ right.' Moreover the circle AIB, having BI an infinitely small chord, is touched by BI at B or C, &c.

Secondly.—If MM and NN are parallels, then QC and PA are parallels, and the point H is at infinity; and therefore the straight line CA lies entirely in its circumference, and gives the point I by its intersection with circle COB or with straight line CB when angle 'θ right' is equal zero.

Thirdly.—If B coincides with C, and that angle θ equals zero, then it is evident that CB may take any direction through C, and therefore that the point I is that in which any straight line through C cuts the circle CHA: hence in this case the circle IAB coincides with AHC, and the points E are identical with the intersections of the circle AHC and straight line MM.

Fourthly.—If B and A coincide, then EAD coincides with EBO, and O coincides with D, and circle CAD with circle CBO. In this case the points I and circles IAB are innumerable, and the problem becomes 'porismatic,' and such that any two lines CO BO making the angle OB right to C $= \theta$ right, will cut MM and NN in E and F, so that P E : Q F :: $m : n$.

Fifthly.—It is to be remarked that there are a great number of other particular cases of this problem, arising from particular relative states of the involved data; but, as the solution is general, it holds good in all cases, although it may sometimes require considerable geometrical address to see the modifications necessary. For instance, MM and NN may be coincident, &c.

DEDUCTIONS.

(The figure for the following deductions to be supplied by the reader.)

THEOREMS.

If P and Q be points in lines MM, NN, and that PB and QC make the angle PB right M equal QC right to N, and that the ratio of PB to QC is as m to n; then H being the point of intersection of PB and QC, the circle CHB is such that D being in point in its circumference, and EF the points in which DB and CD cut MM and NN, we shall have PE : QF :: PB : QC :: $m : n$.

And I being the point of intersection of MM and NN, it is evident a circle can pass through the points IPQH.

And if O be the other point in which these two circles CHB IPQH cut, it is evident the triangles PBO QCO are similar, and that PO has to QO the same ratio PB has to QC, or which OB has to OC, or of m to n.

PORISM.

Given two straight lines MM NN, *and two points* B C, *in position; then* P *being any point arbitrarily assumed in* MM, *a point* Q *in* NN *can be found, and an angular magnitude* 'θ,' *such that* BD *and* CD *being any two lines through* B *and* C *making the angle* DB *right to* C $= \theta$ *right, and* E *and* F *the respective points in which these lines cut* MM *and* NN, *we shall have* PE *to* QF *in a constant determinable ratio.*

The angle θ right is evidently = angle MM right to NN, &c.

PORISM.

Given two points P *and* Q, *and given the two points* BC *in the circumference of a given circle: then* MM *being an arbitrarily assumed line through the point* P, *a line* NN *passing through* Q *can be found, such that* D *being any point in the given circumference, and* EF *those in which* DB *and* DC *cut* MM *and* NN, *we shall have* PE *to* QP *in a constant determinable ratio.*

The ratio is evidently that of PB to QC, &c.

PORISM.

Given two points B C *in a given circle, and a point* P *in a given line* MM; *another straight line* NN *and a point* Q *in it can be found, such that* D *being any point in the given circle, then will* DB *and* DC *cut* MM *and* NN *in points* E *and* F, *making* PE *to* QP *in a given ratio* m : n.

For PBH and HCQ are in position, and the point Q making QC : PB :: $m : n$, and QNN making angle QC right to N = PB right to M.

PORISM.

Let P *and* Q *be given points in the circumference of a circle; if in lines from these points to any point* H *in circumference, we take* PB *and* QC, *having to each other a given ratio, then will the circle* BHC *pass through a fixed determinable point* O *in the given circumferance.*

It is evident O is such that PO : QO :: PB : QC, &c.

And if the points PQ and circle be fixed only, then O is fixed, &c.

PORISM.

Given two points PQ, *in two lines* MM NN, *and given any other point* B; *another point* C *can be found, and a circle through* B *and* C *described, such that* D *being any point in its circumference, and* EF *those in which* DB *and* DC *cut* MM *and* NN, *we shall have* PE *to* QF *in a given ratio of* m *to* n.

For PB is given, and therefore QC making angle QC right to N = PB right to M is determinable in position; and since QC : PB :: $m : n$, the point C is determinable. The point H is the intersection of QC and PB, and therefore the circle CHBD is determinable, &c.

PORISM.

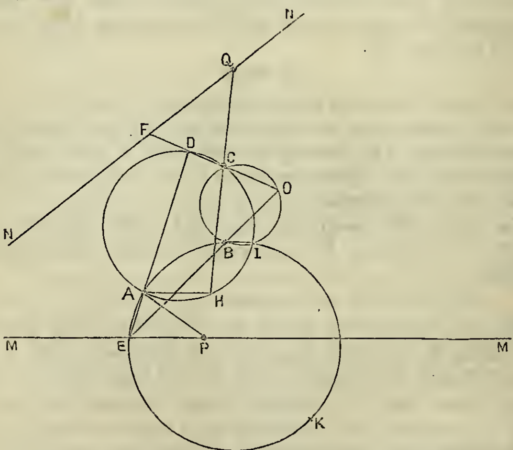
Let HQ *be fixed points, and* H *any point in a fixed circle; a point* O *can be found, such that any circle through it and the point* H *shall cut* PH *and* QH *in* B *and* C, *so that* D *and* I *being any two points in the respective circles,* BHC *and* BHQ, *and* EF *the points in which* DB *and* DC *cut* IP IQ, *then will* PB *have to* QC *and* PE *have to* QF *the given ratio which* m *has to* n.

The point O is evidently in circle PHQ, and PO : QC :: m : n, &c.

PORISM.

Given the points P *and* Q *in given straight lines* MM NN; *a point* C *and angle* θ *right can be found, such that* E *and* F *being any two points in the given lines, such that* CE *right to* F *shall be equal angle* θ *right, then will* PE *have to* QF *a given ratio of* m *to* n.

If we describe a circle through PQ, and the intersection of the given lines, the point C is evidently in its circumference, and such that PC : QC :: m : n. The θ angle is evidently equal MM right to NN.



Given two straight lines MM NN, *and the point* P *in* MM, *and* Q *in* NN; *through two given points* BC, *to draw two straight lines* BO CO, *making an angle* OB *right to* C *of a given magnitude* θ *right, and such that* E *and* F *being the* *points in which* OB *and* OC *respectively cut* MM *and* NN, *we shall have the rectangle under* PE *and* QF *equal to a given rectangle* m . n.

ANALYSIS.

Suppose we draw PA making angle PA right to E equal QF, right to C, and PA . QC equal in sign and magnitude to PE . QF: then it is evident the triangle APE is similar to FQC, and that angle EP left to A is equal CQ left to F; therefore if H be the point in which AH parallel to MM cuts QC, and D the point in which EA cuts FC, it follows that a circle can pass through ACDH. It is also evident that the points AH and circle AHC are given.

And since the angle OB right to C is given in magnitude, the circle OBC is given; and therefore also the point I in which is again cuts circle AHC. Again, the angle AI right to E or D being equal to angle CI right to D or O, it is equal angle BI right to O or E; therefore a circle can pass through AIBE. But AI and B are given points; therefore the circle AIB is given, and hence the point E where it cuts MM, and therefore EBO and COF.

COMPOSITION.

Draw QC; through P draw PA making angle PA right to M equal QN right to C, and on it take PA so that PA.QC shall be equal in sign and magnitude to *m n*; through CA and the point H in which AH parallel to MM cuts QC describe a circle; through B and C describe a circle whose circumference is such that BB and CC being any two lines through B and C to cut on it, we shall have angle BB right to CC equal angle θ right; through the points AB, and the other point I in which this circle cuts circle AHC describe a circle; through either point E in which circle ABI cuts MM, draw EB to cut circle CIB in O; draw OC to cut NN in F: then will EBO and DCO be as required.

For let D be the intersection of EA and FC.

The angle AE or AD right to I equals angle BE or BO right to I = equal CO or CD right to I, therefore the point D lies in circumference AHC.

The angle AD right to H or EA right to P = CD or CF right to H or Q; and, therefore, since angle PE right to A = QC right to F, the triangles PAE, QCF, give us PE·QF = PA·QC = *m·n*.

DISCUSSION.

If the signs of *m* and *n* be given, and that the directions on the given lines are specified, then there is but one answerable point A, one circle AIB, two points E real or unreal, and therefore but two solutions. But if *m.n* be not restricted as to sign, there are then two points A, two circles AIB, and therefore four answerable points E and ∴ four solutions. Moreover, since the two points A must be on opposite sides of MM, it follows that one of the circles AIB will always cut MM in two real points E.

Limiting values for the angular magnitude 'θ right.'

To find the limiting values of the angle 'θ right' when the rest of the data is unchangeable, we may proceed as follows:—Thus, looking on the triangle DEO, we see the angle DO right to E is constant; and it is evident that when OE right to D or OB right to C is at its limit, then must ED right to O or EA right to B be at its limit. But this last angle is evidently at a limit when the circle BAE touches MM. Hence it is evident that by describing the two circles through A and B which touch MM, and putting *i* and *i* for the other points in which they cut circle ACH, then will the angles *i*B right to C, and *i*B right to C, be the required limits. And it is, moreover, evident that according as the magnitude of any angle 'θ right' is not comprehended between these limits, or equals one of them, or is comprehended between them, so accordingly will the corresponding circle AIB cut MM in two imaginary, in two real and coincident, or in two real and distinct points E.

Limiting values for m.n.

The limiting values of *m.n*, the rest of the data being unchangeable, may be found as follows:—

It is evident the point A varies on PA according as the values of *m.n* vary, and that when *m.n* is at its limits, the points A will be in limiting positions on PA. Now it is, moreover, evident that the other point K in which circle AEB cuts PA is fixed and given, no matter how A may vary on PA (because angle KA right to B = EA or ED right to B or O, and that the angles OB right to C and DC right to A, or which is the same thing, because the angles OE right to D and DO right to E are given in magnitudes). But the points A are in their limiting positions when the circles AEBK are in limiting positions or touching MM. Hence putting *a* and *a* for the other two points in which the circles through B and K to touch MM again cut PA, we shall have

Pa.QC and Pa.QC the required limits. And it is evident these limiting values have like signs, and that according as any value of $m.n$ (of the same sign) is comprehended between them, equal to one of them, or not comprehended between, so accordingly will the corresponding circle AIKB cut MM in two imaginary, in two real and coincident, or in two real and distinct points, E, &c. If B and K be not on the same side of MM, the limits are imaginary, and therefore the points E always real.

Remarks on the consequences arising from supposing particular positions and magnitudes for the involved data.

First.—If the point B were coincident with C, the rest being as general as may be desired. Here the circle COB is infinitely small, and the points I and O are coincident with C. It is also evident that the direction CI is, in this case, on a tangent to the circle AHC at C, and that the direction of BI is determinable, because angle IC right to B = angle θ right. It is also evident the circle AIB, having the zero chord IB, touches the direction BI in B, and is therefore determinable, &c.

Secondly.—If B coincides with C, and that angle θ right = zero, the rest being as general as may be. In this case it is evident that CB may take any direction through C, and therefore that the point I is that in which any straight line through C cuts circle CHA. Hence circle IAB coincides with AHC, and the points E are those in which the circle AHC intersects MM.

Thirdly.—If MM and CQ are parallels, then QC and HA are parallels, and the point H is at infinity, and therefore the straight line CA lies entirely in the circumference CAH, and gives the point I by its intersection with circle CBO. And it may be further remarked that should the angle θ right = zero, then will the straight line CB lie wholly in circumference CBO, and give the point I by its intersection with CA.

Fourthly.—If the points I and B coincide, the circles AIB, CBO, touch in B. And if I and C coincide, the circles AHC, CBO, touch in C.

Fifthly.—If the points B and A coincide, then the straight lines EBO EAD coincide, and the point O coincides with D, and the circle CBO with CAD. In this case the points I are evidently innumerable, and occupy all positions in the circumference CHB, and the problem becomes 'porismatic,' and such that any two straight lines CO and BO, making angle OB right to C = θ right, will cut MM and NN in E and F, so that PE · QF = PA · QC = $m \cdot n$.

Sixthly.—There are scores of other particular cases, and as the solution is general, the modifications for these cases may be easily made, and will afford useful exercise.

DEDUCTIONS.

(Figures for the following to be made by the reader.)

In the case of this problem, in which B coincides with A, we have angle PB right to M = QN right to C; and, therefore, putting p and q for the respective points in which QC and PB cut MM and NN, it is evident a circle can pass through PQ pq. And we have the line BH parallel to MM, the point H being in circle BOC. And L being the other point in which PB cuts circle BOC, it is evident CL is parallel to NN.

Hence we have the following theorem:—Let PQ pq be any four points in the circumference of any circle; if on Pq and Qp we take any two points BC, so that PB · QC shall be of a constant magnitude $m \cdot n$, and draw BH and CL parallels to Pp and Qq, to cut Qq and Pp in H and L; then will the points BCHL be in the circumference of a circle; and O being any point whatever in this circumference, the lines OB OC will cut Pp and Qq in E and F, so that PE · QF = PB · QC = $m \cdot n$. And the following deductions are obvious.

PORISM.

Given two straight lines MM NN, *and two points* BC *in position and an angular magnitude 'θ right;' two points* PQ *can be found, one in each of the given lines, such that* BO *and* CO *being any two lines inflected, making angle* OB *right to* C = θ *right, and* E *and* F *the points in which these lines cut* MM *and* NN *respectively; we shall have* PE.QF *of a constant determinable magnitude.*

For the circle BOC is determinable, and BH and CL, and therefore BL and CH, and the points P and Q, in which these last cut MM and NN. And we have PE · QF always equal PB · QC.

PORISM.

Given an angular magnitude θ right, and two points BC *through two other given points* P *and* Q, *two lines* MM *and* NN, *and but two can be drawn, such that* BO *and* CO *being* *any two lines making the angle* OB *right to* C $= \theta$ *right, and cutting* MM *and* NN *in* E *and* F, *we shall have* PE · QF = *a constant determinable magnitude.*

For the circle BCO is determinable, as also PB, QC, and points L and H, and therefore MM and NN, through P and Q, parallels to BH and CL. And we have the rectangle PE · QF equal PB · QC.

PORISM.

Given the points P *and* Q *in the given straight lines* MM *and* NN, *and given any other point* B *in the same plane; a point* C *may be found, and an angular magnitude* θ *right, such that* BO *and* CO *being any two lines making angle* OB *right to* C $= \theta$ *right, the points* E *and* F, *in which they cut* MM *and* NN, *are so related that* PE · QF *shall be of any given magnitude* m · n.

For the line PBq is determinable, and ∴ circle PqQ, and hence point p; and, since PB · QC must be equal m . n, the point C is determinable. The angle θ right is evidently equal PM right to B.

PORISM.

Given an angle θ *right, and the points* P *and* Q *in given lines* MM NN; *two points* BC *may be found on any curve or line, such that* BO *and* CO *making the angle* OB *right to* C = θ *right, and cutting the given lines in* E *and* P, *we shall* PE . QF *of a constant determinable magnitude.*

For the angle θ right being given, Pq and Qp are determinable, and the points B and C in which they cut the given curve or line, and therefore PB . QC, or the magnitude of PE BF.

The points BC or one of them may be sometimes imaginary, &c.

PORISM.

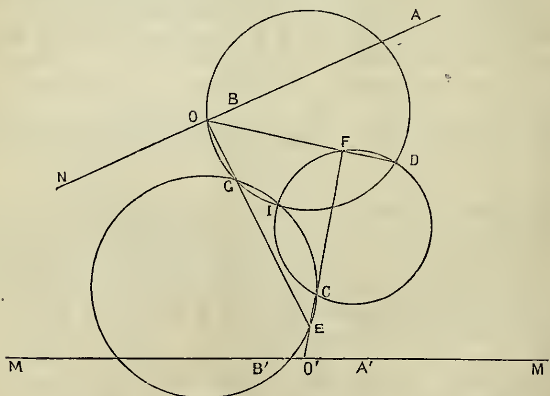
Given the straight lines MM NN, *and the point* P *and* Q *therein; two points* B *and* C *may be found in the circumference of a given circle, such that* O *being any other point whatever in the circumference, the lines* OB *and* OC *shall cut* MM *and* LL *in points* E *and* F *so that* PE . QF *will be of a constant determinable magnitude.*

Here we have to draw PBL and QCH so that PH and QL may be parallels to MM and NN respectively. The points B and C thus determined give PE . QF = PB . QC.—The method of drawing PBL and QCH as indicated is obvious. For drawing LS parallel to PH or MM to cut circle in S, and then SH to cut MM in R, it is evident, P and R are equidistant from the centre of the circle, and therefore that R is determinable. The point T where MM cuts NN is also. And the angle HC right to S being equal, LC right to S is equal NN right to MM; and therefore a circle can pass through TRH and Q. Hence the point H where this circle cuts the given one is determinable, and hence HB parallel to MM, and point B, line HQ and point C.

PORISM.

Given the points P *and* Q *in given straight lines* MM NN, *a point* C *and an angle 'θ right' can be found such that* E *and* F *being any two points making* PE · QF *equal a given magnitude* m · n, *the angle* CE *right to* F *shall be equal angle θ right.*

It is evident that the determination of the point C involves the solution of the following problem :—Given the base PQ, the difference of the angles at the base, and the rectangle under the sides of the triangle PQC to construct the triangle. It may also be remarked that when MM and NN coincide, the difference of the angles at the base is zero; and when moreover $m \cdot n$ is negative and greater than one-fourth of PQ . QP, then we have in an implicit manner the theorem which Chasles establishes concerning homographic divisions on the same line whose double points are imaginary. See Chesles' Géométric Superièure, page 118.



Given a pair of homographic divisions on two straight lines MM NN, *and likewise a second pair of other homographic* *divisions on these same two lines; to find two points* OO′, *one in each line, such that they shall be corresponding points in the two pairs of homographic divisions.*

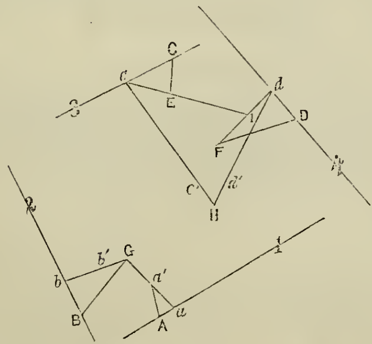
ANALYSIS.

According as the first pair of homographic divisions cut the lines MM NN into proportional segments or not, find a pair of points AA′ in these lines, such that AO shall have to A′O′ a constant determinable ratio, or find the two points A and A′ so that AO . A′O′ shall be a constant determinable rectangle. And according as the second pair of homographic divisions cut MM NN into proportional segments or not, find the points B and B in these lines such that BO and B′O′ shall have a constant determinable ratio to each other, or that BO . B′O′ shall be of a constant determinable magnitude.

Now since the points AA′ are given, if we assume a point *c*, then (by one of the preceding porisms) we can find a point D and circle DFC such that OD and O′C will cut in some point F in circle DFC.

And since B & B′ are given points, we can find another point G and circle GCE (by one of the preceding porisms) such that OG & O′C will cut in some point E in circle GCE.

Now since angle GI right to E or O=CI right to E or F=DI right to F or O, therefore a circle can pass through GID and O; but GDI are given points, therefore the point O in which this circle GDI cuts NN is given, and hence OGE and CEO′ and point O′ in MM.



Given the lines 1, 2, 3, 4, *and the points* A, B, C, D, *one in each line; given also the points* a′, b′, c′, d′; *through* a′ *and* b′

to draw a′G b′G, *making angle* Gb′ *right to* a′ *of a given magnitude, and through* c′ *and* d′ *draw* c′H *and* d′H *making angle* Hd′ *right to* c′ *of a given magnitude, so that* a, b, c, d *being the points in which* a′G, b′G, c′H, d′H *cut the straight lines* 1, 2, 3, 4, *we shall have* Aa *to* Dd *in a given ratio, and the rectangle under* Bb *and* Cc *of a given magnitude.*

ANALYSIS.

Draw a'A b'B. Suppose we draw CE making angle CE right to c = to angle Bb right to b', and CE·Bb' = Cc·Bb; then E is given, and we have also the angle cC right to E = b'B right to b or G.

Again, drawing DF, making angle DF right to d = angle Aa' right to a, and DF to Aa' as Dd is to Aa; then F is a given point, and the angle FD right to d is = angle a'A right to a or G.

Now EF are given points, and FD, Cc given lines in position; and $a'b'$ are given points, and a'A b'B lines given in position; hence as G is in the circumference of a given circle $b'a'$G, it follows (from improved theorems) that the intersection I of cE and dF is in the circumference of a determinable circle through E and F.

But (by porism No. 1 of second problem) we can find points P and Q in lines 3 and 4, corresponding to circle EFI, such that Pc.Qd shall be of a constant determinable magnitude; similarly we can find points M and N in the same lines corresponding to the given circle c'Hd', such that the rectangle Mc.Nd shall be of a constant determinable magnitude: therefore by problem 3rd we can find the points c and d.

REMARK.

If instead of the ratio Aa to Dd we were given the rectangle under them, or that instead of the rectangle under Bb and Cc we were given their ratio, the method of solution is obvious from the above method of proceeding.